

# ON HOMOLOGY 3-SPHERES DEFINED BY TWO KNOTS

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**ABSTRACT.** We show that if each of  $K_1$  and  $K_2$  is a trefoil knot or a figure eight knot, the homology 3-sphere defined by the Kirby diagram which is a simple link of  $K_1$  and  $K_2$  with framing  $(0, n)$  is represented by an  $n$ -twisted Whitehead double of  $K_2$ .

## 1. Introduction

We define  $W_n(K_1, K_2)$  to be the 4-dimensional handlebody represented by the following Kirby diagram, and define  $M_n(K_1, K_2)$  to be  $\partial(W_n)$ , where  $K_1$  and  $K_2$  are knots. Note that  $M_n(K_1, K_2)$  is a homology 3-sphere.



FIGURE 1.1.  $W_n(K_1, K_2)$

When  $K_1$  and  $K_2$  are right handed trefoil knots  $T_{2,3}$ , Y. Matsumoto asked in [5] whether  $M_0(T_{2,3}, T_{2,3})$  bounds a contractible 4-manifold or not. By Gordon's result [3], if  $n$  is odd,  $M_n(T_{2,3}, T_{2,3})$  does not bound any contractible 4-manifold. If  $n$  is 6, N. Maruyama [6] proved that  $M_6(T_{2,3}, T_{2,3})$  bounds a contractible 4-manifold. If  $n$  is 0, S. Akbulut [1] proved that  $M_0(T_{2,3}, T_{2,3})$  does not bound any contractible 4-manifold.

In this note, we show that if each of  $K_1$  and  $K_2$  is a trefoil knot or a figure eight knot, the homology 3-sphere defined by Figure 1.1 is represented by an  $n$ -twisted Whitehead double of  $K_2$ .

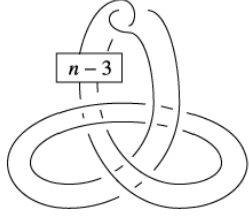
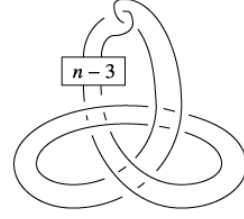
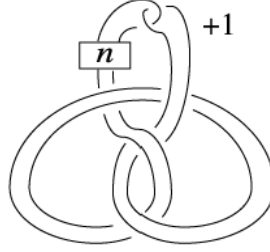
### Notations.

(i). Let  $K$  be a knot, we define  $D_+(K, n)$  ( or  $D_-(K, n)$  ) to be the  $n$ -twisted Whitehead double of  $K$  with a positive hook ( or a negative hook ). For example, when  $K$  is a right handed trefoil knot  $T_{2,3}$ ,  $D_+(T_{2,3}, n)$  is the knot represented by Figure 1.2, and  $D_-(T_{2,3}, n)$  is the knot represented by Figure 1.3.

(ii). We define  $S_{\pm 1}^3(K)$  to be the  $\pm 1$ -surgery along a knot  $K$ . For example, when  $K$  is a figure eight knot,  $S_{+1}^3(D_+(K, n))$  is represented by Figure 1.4.

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FIGURE 1.2.  $D_+(T_{2,3}, n)$ FIGURE 1.3.  $D_-(T_{2,3}, n)$ FIGURE 1.4.  $S^3_{+1}(D_+(K, n))$ 

**Theorem 1.1.** *If each of  $K_1$  and  $K_2$  is a trefoil knot or a figure eight knot,  $M_n(K_1, K_2)$  is represented by the second column on the following table.  $\lambda(S^3_{\pm 1}(D_{\pm}(K, n)))$  is the Casson invariant of  $S^3_{\pm 1}(D_{\pm}(K, n))$ .*

$M_n(K_1, K_2)$	$S^3_{\pm 1}(D_{\pm}(K, n))$	$\lambda(S^3_{\pm 1}(D_{\pm}(K, n)))$
$K_1 : \text{right handed trefoil}, K_2 : \text{right handed trefoil}$	$S^3_{+1}(D_+(K_2, n))$	$-n$
$K_1 : \text{left handed trefoil}, K_2 : \text{right handed trefoil}$	$S^3_{-1}(D_-(K_2, n))$	$-n$
$K_1 : \text{figure eight knot}, K_2 : \text{right handed trefoil}$	$S^3_{-1}(D_+(K_2, n))$	$n$
	$\cong S^3_{+1}(D_-(K_2, n))$	
$K_1 : \text{right handed trefoil}, K_2 : \text{figure eight knot}$	$S^3_{+1}(D_+(K_2, n))$	$-n$
$K_1 : \text{figure eight knot}, K_2 : \text{figure eight knot}$	$S^3_{+1}(D_-(K_2, n))$	$n$

We will prove Theorem 1.1 in Section 2.

**Remark.** When  $n$  is 0, S. Akbulut [1] shows essentially the same result of the first row on the table by a different method.

**Corollary 1.2** (Gordon [3], cf. Y. Matsumoto [7] §3.1.). *Let  $M_n(K_1, K_2)$  be one of the manifolds in the above table. If  $n$  is odd,  $M_n(K_1, K_2)$  does not bound any contractible 4-manifold.*

*Proof.* A short proof of this result goes as follows:

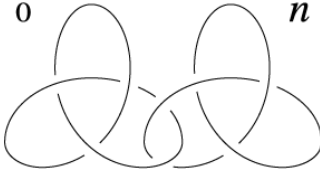
The Casson invariant, when reduced modulo 2, is the Rohlin invariant:

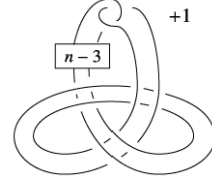
$$\lambda(M_n(K_1, K_2)) \equiv \mu(M_n(K_1, K_2)) \pmod{2}$$

By Theorem 1.1,  $\lambda(M_n(K_1, K_2))$  is  $n$  or  $-n$ . Therefore if  $n$  is odd, we have  $\mu(M_n(K_1, K_2)) \equiv 1 \pmod{2}$ , and so  $M_n(K_1, K_2)$  does not bound any contractible 4-manifold.  $\square$

**Corollary 1.3** (N. Maruyama [6]). *If  $K_1$  and  $K_2$  are right handed trefoil knots  $T_{2,3}$ ,  $M_6(T_{2,3}, T_{2,3})$  bounds a contractible 4-manifold.*

*Proof.* By the first row on Theorem 1.1's table,  $M_n(T_{2,3}, T_{2,3})$  is represented by  $S_{+1}^3(D_+(T_{2,3}, n))$ . If  $n$  is 6,  $D_+(T_{2,3}, 6)$  is known to be a slice knot ([8], p226). Therefore by [3],  $M_6(T_{2,3}, T_{2,3})$  bounds a contractible 4-manifold.

FIGURE 1.5.  $M_n(T_{2,3}, T_{2,3})$ 

$$\cong$$
FIGURE 1.6.  $S_{+1}^3(D_+(T_{2,3}, n))$ 

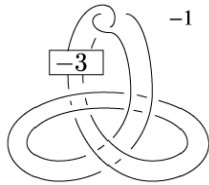
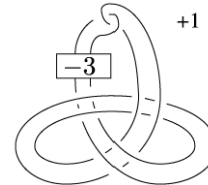
$$\square$$

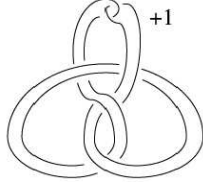
**Corollary 1.4.** *If  $n$  is 0,  $D_+(T_{2,3}, 0)$  is not a slice knot.*

*Proof.* By [1],  $M_0(T_{2,3}, T_{2,3})$  does not bound any contractible 4-manifold. Therefore  $D_+(T_{2,3}, 0)$  is not a slice knot.  $\square$

**Remark.** M. Hedden [4] showed that if  $n$  is smaller than 2,  $D_+(T_{2,3}, n)$  is not a slice knot.

**Corollary 1.5.** *Let  $T_{2,3}$  be a right handed trefoil knot and  $4_1$  be a figure eight knot. The homology 3-spheres  $S_{-1}^3(D_+(T_{2,3}, 0))$ ,  $S_{+1}^3(D_-(T_{2,3}, 0))$  and  $S_{+1}^3(D_+(4_1, 0))$  are pairwise diffeomorphic.*

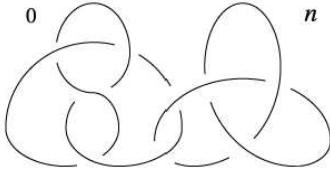
FIGURE 1.7.  $S_{-1}^3(D_+(T_{2,3}, 0))$ FIGURE 1.8.  $S_{+1}^3(D_-(T_{2,3}, 0))$

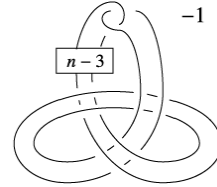
FIGURE 1.9.  $S^3_{+1}(D_+(4_1, 0))$ 

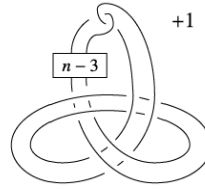
*Proof.* By the third row and the fourth row on Theorem 1.1's table, if  $n = 0$ , the 4-dimensional handlebodies defined by Figures 1.7, 1.8 and 1.9 have the same boundaries. Therefore the homology 3-spheres  $S^3_{-1}(D_+(T_{2,3}, 0))$ ,  $S^3_{+1}(D_-(T_{2,3}, 0))$  and  $S^3_{+1}(D_+(4_1, 0))$  are pairwise diffeomorphic.  $\square$

**Corollary 1.6.** *If  $K_1$  is a figure eight knot and  $K_2$  is a right handed trefoil knot (see Figure 1.10), then  $M_6(K_1, K_2)$  bounds a contractible 4-manifold.*

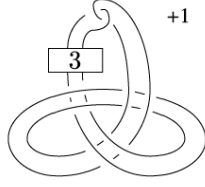
*Proof.* By the third row on Theorem 1.1's table,  $M_n(K_1, K_2)$  is represented by  $S^3_{-1}(D_+(K_2, n))$  and also by  $S^3_{+1}(D_-(K_2, n))$ . If  $n$  is 6,  $D_+(K_2, 6)$  is known to be a slice knot ([8], p226). Therefore by [3],  $M_6(K_1, K_2)$  bounds a contractible 4-manifold.

FIGURE 1.10.  $M_n(K_1, K_2)$ 

$$\cong$$
FIGURE 1.11.  $S^3_{-1}(D_+(K_2, n))$ 

$$\cong$$
FIGURE 1.12.  $S^3_{+1}(D_-(K_2, n))$ 
 $\square$ 

**Remark.** By Corollary 1.6, the homology 3-sphere  $S^3_{+1}(D_-(T_{2,3}, 6))$  bounds a contractible 4-manifold. The author does not know whether the knot  $D_-(T_{2,3}, 6)$  is a slice knot or not.

FIGURE 1.13.  $S^3_{+1}(D_-(T_{2,3}, 6))$ 

**Question.** Let  $V_n^1$  be the 4-dimensional handlebody defined by Figure 1.12, and  $V_n^2$  be the 4-dimensional handlebody defined by Figure 1.11. Since  $\partial(V_n^1)$  is diffeomorphic to  $\partial(V_n^2)$  by Theorem 1.1, we have a closed 4-manifold  $V_n^1 \cup_{\partial} (-V_n^2)$ . Because  $D_+(T_{2,3}, 6)$  is a slice knot, we have a smooth  $S^2$  with self intersection  $-1$  in  $V_6^2$  representing a generator of  $H_2(V_6^2)$ . Blow down this smooth  $S^2$  from the  $V_6^1 \cup_{\partial} (-V_6^2)$ . Then we are left with a closed smooth 4-manifold homotopy equivalent to  $\mathbb{CP}^2$ . Is this 4-manifold diffeomorphic to  $\mathbb{CP}^2$ ?

**Proposition 1.7.**  $V_n^1 \cup_{\partial} (-V_n^2)$  is diffeomorphic to  $\mathbb{CP}^2 \# \mathbb{CP}^2$ .

We show this fact in Section 3.

It seems that Theorem 1.1 is related to [6] Corollary 8 (3), but the author could not understand the relationship clearly.

The author does not know whether there is an even number  $n \neq 0, 6$ , such that  $M_n(T_{2,3}, T_{2,3})$  bounds a contractible 4-manifold or not. M. Tange [10] proved that if  $n$  is smaller than 2,  $M_n(T_{2,3}, T_{2,3})$  does not bound any contractible 4-manifold by computing the Heegaard Floer homology  $HF^+(M_n(T_{2,3}, T_{2,3}))$  and the correction term  $d(M_n(T_{2,3}, T_{2,3}))$ .

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## 2. Proof of Theorem 1.1

In this section, first we show that  $M_n(K_1, K_2)$  is represented by  $S^3_{\pm 1}(D_{\pm}(K_2, n))$ . Next we compute the Casson invariant  $\lambda(S^3_{\pm 1}(D_{\pm}(K_2, n)))$ .

**2(i). Proof of the first row on Theorem 1.1's table.**  $K_1$  and  $K_2$  are right handed trefoil knots.

*Proof.* We show that the 4-manifolds represented by Figures 2.1 and 2.17 have the same boundaries by following Kirby Calculus:

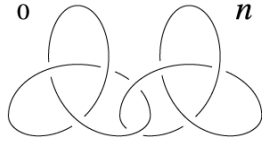
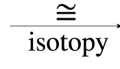
FIGURE 2.1.  $M_n(K_1, K_2)$ 

FIGURE 2.2.

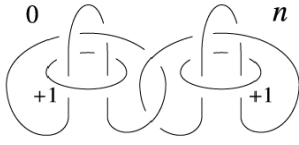


FIGURE 2.3.

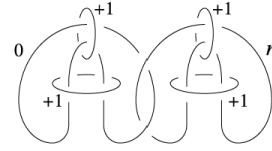


FIGURE 2.4.

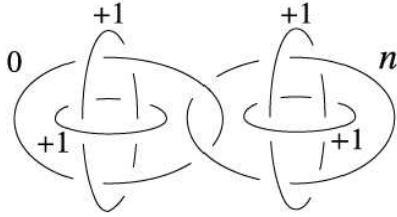
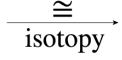


FIGURE 2.5.

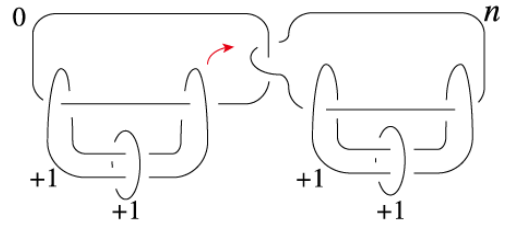
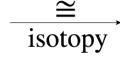


FIGURE 2.6.

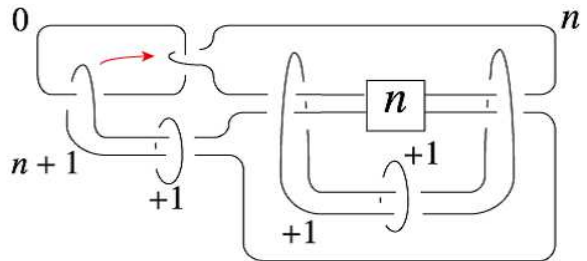
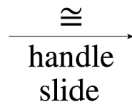


FIGURE 2.7.

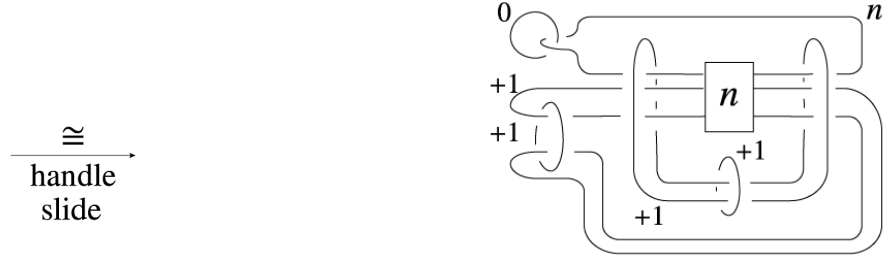


FIGURE 2.8.

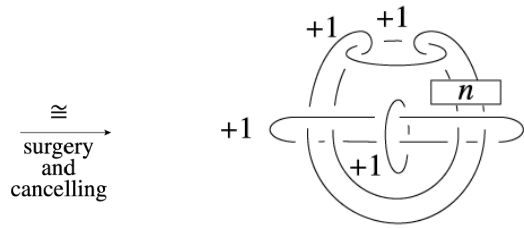


FIGURE 2.9.

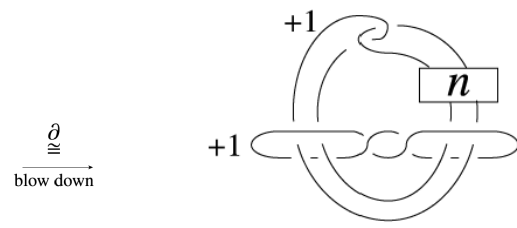


FIGURE 2.10.

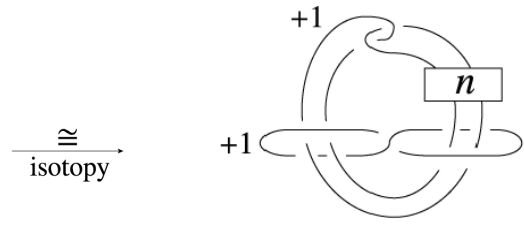


FIGURE 2.11.

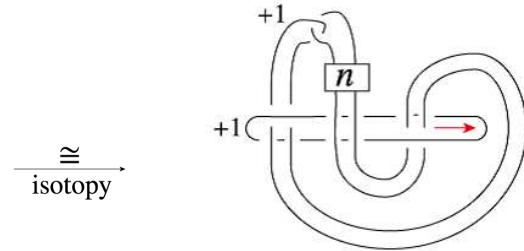


FIGURE 2.12.

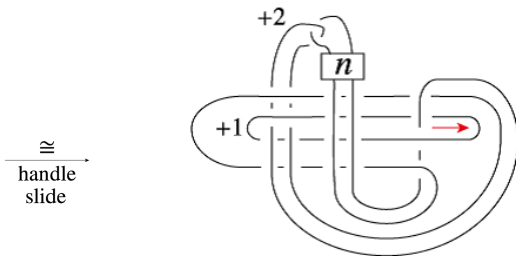


FIGURE 2.13.

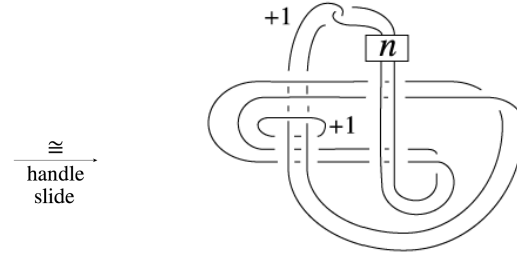


FIGURE 2.14.

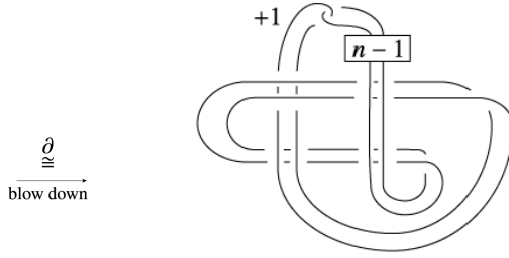


FIGURE 2.15.

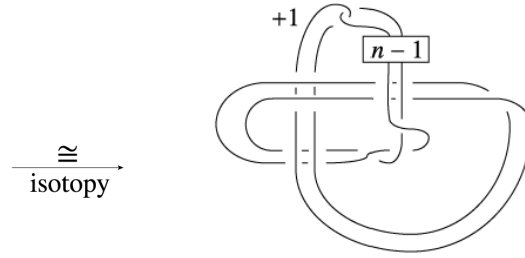


FIGURE 2.16.

FIGURE 2.17.  $S^3_{+1}(D_+(K_2, n))$ 

□

2(ii). **Proof of the second row on Theorem 1.1's table.**  $K_1$  is a left handed trefoil knot and  $K_2$  is a right handed trefoil knot.

*Proof.* We show that the 4-manifolds represented by Figures 2.18 and 2.23 have the same boundaries.

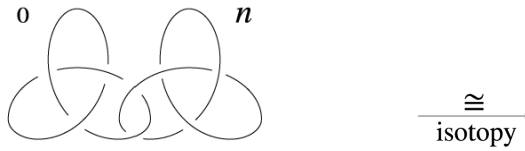
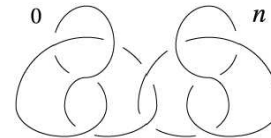
FIGURE 2.18.  $M_n(K_1, K_2)$ 

FIGURE 2.19.

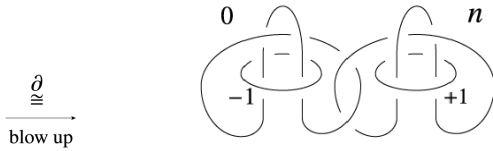


FIGURE 2.20.

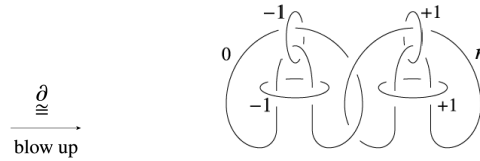


FIGURE 2.21.



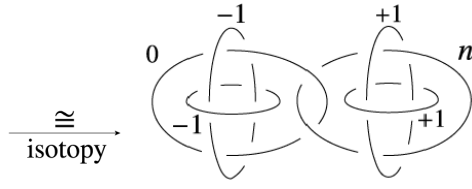
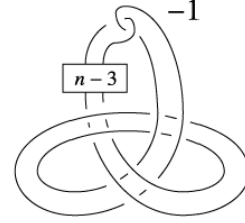


FIGURE 2.22.

$\partial \cong$   
by the same  
process except for  
the sign of  
the framing  
FIGURE 2.5 - FIGURE 2.17

FIGURE 2.23.  $S_{-1}^3(D_-(K_2, n))$ 

□

2(iii). **Proof of the third row on Theorem 1.1's table.**  $K_1$  is a figure eight knot and  $K_2$  is a right handed trefoil knot.

*Proof.* We show that the 4-manifolds represented by Figures 2.24, 2.29 and 2.35 have the same boundaries.

FIGURE 2.24.  $M_n(K_1, K_2)$ 

$\cong$   
isotopy



FIGURE 2.25.

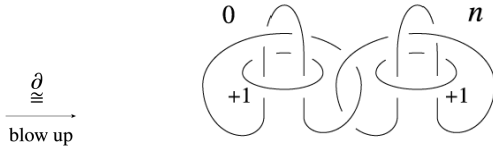


FIGURE 2.26.

$\partial \cong$   
blow up

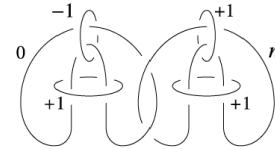


FIGURE 2.27.

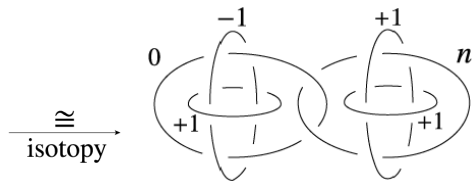


FIGURE 2.28.

$\partial \cong$   
by the same  
process except for  
the sign of  
the framing  
FIGURE 2.5 - FIGURE 2.17

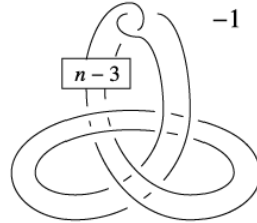
FIGURE 2.29.  $S_{-1}^3(D_+(K_2, n))$

Figure 2.30 is the same diagram of Figure 2.24, but by using the invertibility of the figure eight knot, we can show that they can be represented by a different doubled knot.

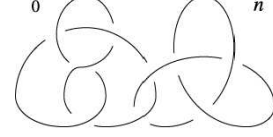
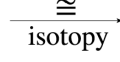
FIGURE 2.30.  $M_n(K_1, K_2)$ 

FIGURE 2.31.



FIGURE 2.32.



FIGURE 2.33.

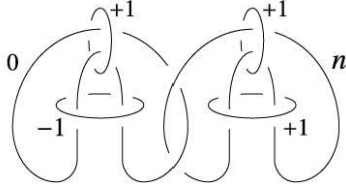
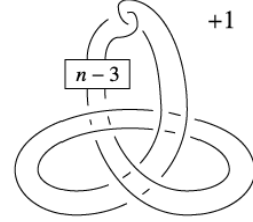
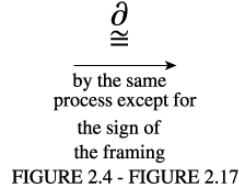


FIGURE 2.34.

FIGURE 2.35.  $S_{+1}^3(D_-(K_2, n))$ 

□

2(iv). **Proof of the fourth row on Theorem 1.1's table.**  $K_1$  is a right handed trefoil knot and  $K_2$  is a figure eight knot.

*Proof.* We show that the 4-manifolds represented by Figures 2.36 and 2.47 have the same boundaries.

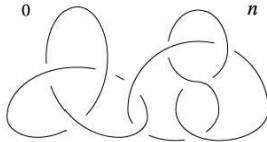
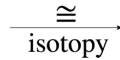
FIGURE 2.36.  $M_n(K_1, K_2)$ 

FIGURE 2.37.

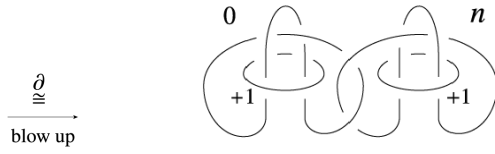


FIGURE 2.38.

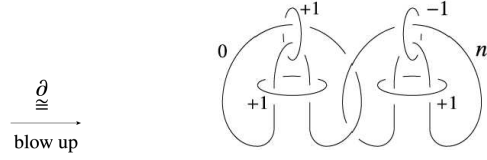


FIGURE 2.39.

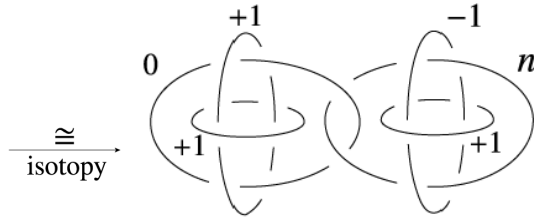


FIGURE 2.40.

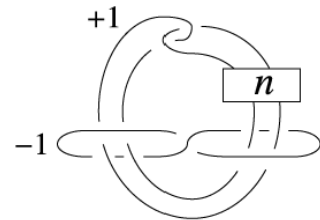
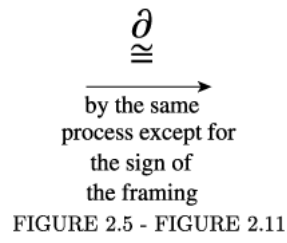


FIGURE 2.41.

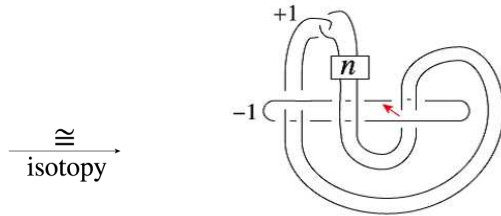


FIGURE 2.42.

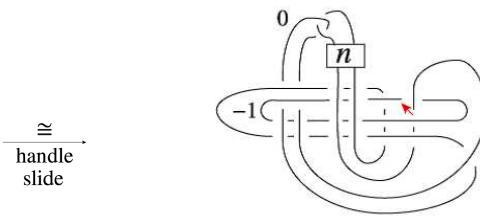


FIGURE 2.43.

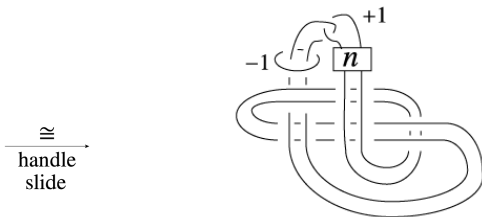


FIGURE 2.44.

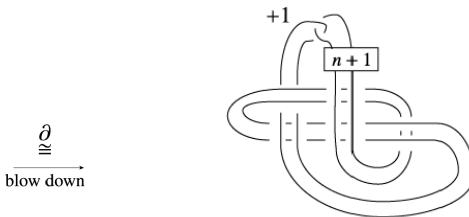


FIGURE 2.45.

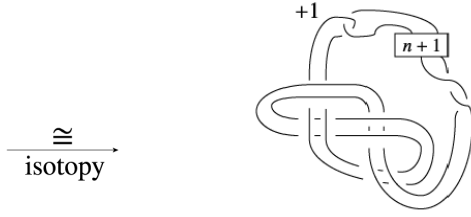
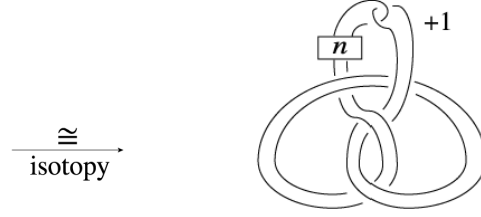


FIGURE 2.46.

FIGURE 2.47.  $S_{+1}^3(D_+(K_2, n))$ 

□

2(v). **Proof of the fifth row on Theorem 1.1's table.**  $K_1$  and  $K_2$  are figure eight knots.

*Proof.* We show that the 4-manifolds represented by Figures 2.48 and 2.55 have the same boundaries.

FIGURE 2.48.  $M_n(K_1, K_2)$ 

FIGURE 2.49.



FIGURE 2.50.

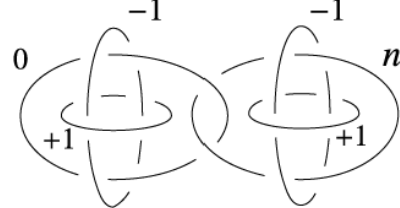
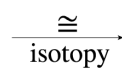
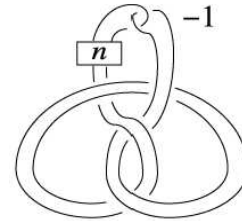
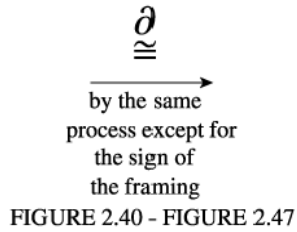


FIGURE 2.51.

FIGURE 2.52.  $S_{-1}^3(D_+(K_2, n))$

Since Figure 2.53 is the same diagram as Figure 2.51, we can show that they can be represented by a different double knot.

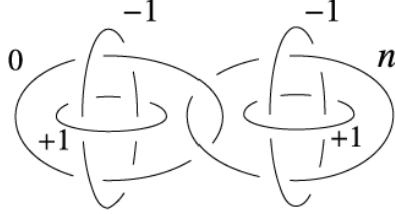


FIGURE 2.53.

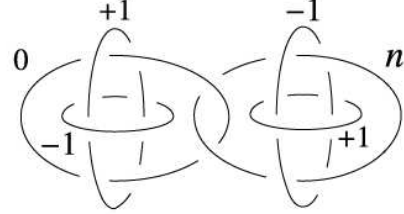
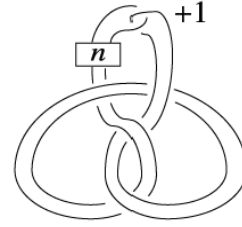
$$\xrightarrow[\text{isotopy}]{\cong}$$


FIGURE 2.54.

$\xrightarrow[\text{by the same process except for the sign of the framing}]{\partial \cong}$   
 FIGURE 2.40 - FIGURE 2.47

FIGURE 2.55.  $S^3_{+1}(D_-(K_2, n))$ 

□

Next we compute the Casson invariant  $\lambda(M_n(K_1, K_2))$ . Now suppose that  $K_-$ ,  $K_+$  and  $K_0$  are links in  $S^3$  which have projections which differ at a single crossing of  $K_-$  as depicted below.

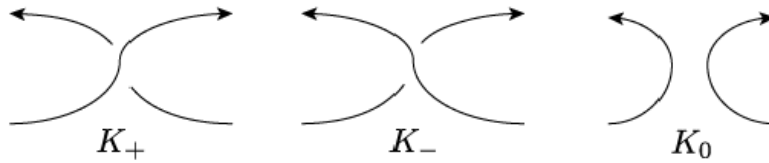


FIGURE 2.56.

**Remark.** Our convention in Figure 2.56 is different from that in [2]. In fact, their  $K_+$  (resp.  $K_-$ ) is our  $K_-$  (resp.  $K_+$ ). We adopt our convention as Figure 2.56 because by our convention  $\lambda'(T_{2,3})$  is computed to be 1, where  $T_{2,3}$  is a right handed trefoil knot. While by their convention  $\lambda'(T_{2,3})$  is computed to be  $-1$ , contradicting the normalization  $\lambda'(T_{2,3}) = 1$  ([2] p148, [9] p52).

**Lemma 2.1** (see [2], p. 143). *Let  $K_-$  be a knot in  $S^3$ . Let  $K_+$  and  $K_0$  be as above. Then  $K_0$  is a two component link and:*

$$\lambda'(K_+) - \lambda'(K_-) = lk(K_0)$$

where  $\lambda'(K)$  is the Casson invariant of a knot  $K$ .

**Lemma 2.2** (Surgery formula, see [9], p. 52). *Let  $K$  be a knot in  $S^3$ . The Casson invariant  $\lambda(S^3_{+1}(K))$  is equal to  $\lambda'(K)$ .*

By Lemmas 2.1, 2.2 and the second column on Theorem 1.1's table, we can compute the Casson invariant  $\lambda(M_n(K_1, K_2))$ .

**2(vi). The Casson invariant of the first row on Theorem 1.1's table.**

*Proof.*  $K_1$  and  $K_2$  are right handed trefoil knots. By 2(i),  $M_n(K_1, K_2)$  is diffeomorphic to  $S^3_{+1}(D_+(K_2, n))$ . Therefore  $\lambda(M_n(K_1, K_2))$  is equal to  $\lambda(S^3_{+1}(D_+(K_2, n)))$ . By Lemmas 2.1 and 2.2, we can compute the Casson invariant  $\lambda(S^3_{+1}(D_+(K_2, n)))$  as follows:

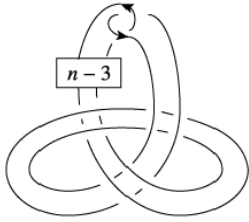


FIGURE 2.57.  $K_+$

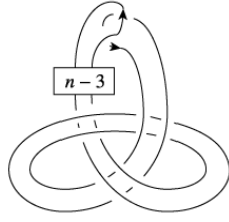


FIGURE 2.58.  $K_-$

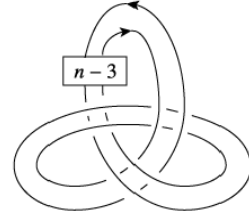
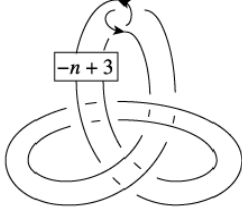
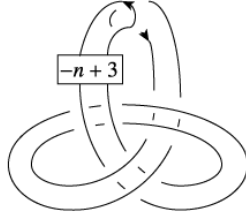
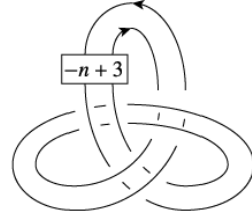


FIGURE 2.59.  $K_0$

By Lemma 2.1,  $\lambda'(K_+) - \lambda'(K_-) = lk(K_0)$ . Since  $K_-$  is a trivial knot,  $\lambda'(K_-) = 0$ .  $lk(K_0)$  is  $-n$ . Therefore,  $\lambda'(K_+) = -n$ . Then  $\lambda(M_n(K_1, K_2)) = \lambda(S^3_{+1}(D_+(K_2, n))) = \lambda'(K_+) = -n$ . □

**2(vii). The Casson invariant of the second row on Theorem 1.1's table.**

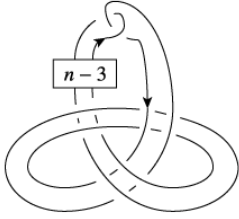
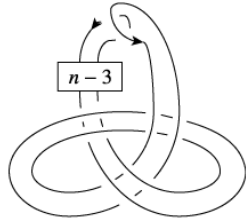
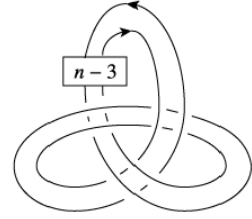
*Proof.*  $K_1$  is a left handed trefoil knot and  $K_2$  is a right handed trefoil knot. By 2(ii),  $M_n(K_1, K_2)$  is diffeomorphic to  $S^3_{-1}(D_-(K_2, n))$ . Therefore  $\lambda(M_n(K_1, K_2))$  is equal to  $\lambda(S^3_{-1}(D_-(K_2, n)))$ . Since  $\lambda(S^3_{-1}(D_-(K_2, n))) = -\lambda(S^3_{+1}(D_+(K_1, -n)))$  (see [9], p. 52, Theorem 3.1.), we compute the Casson invariant  $\lambda(S^3_{+1}(D_+(K_1, -n)))$  as follows:

FIGURE 2.60.  $K_+$ FIGURE 2.61.  $K_-$ FIGURE 2.62.  $K_0$ 

By Lemma 2.1,  $\lambda'(K_+) - \lambda'(K_-) = lk(K_0)$ . Since  $K_-$  is a trivial knot,  $\lambda'(K_-) = 0$ .  $lk(K_0)$  is  $n$ . Therefore,  $\lambda'(K_+) = n$ . Then  $\lambda(M_n(K_1, K_2)) = \lambda(S_{+1}^3(D_-(K_2, n))) = -\lambda(S_{+1}^3(D_+(K_1, -n))) = -\lambda'(K_+) = -n$ .  $\square$

2(viii). **The Casson invariant of the third row on Theorem 1.1's table.**

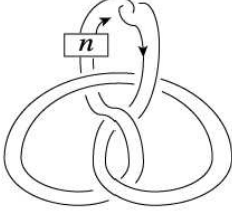
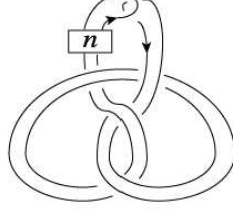
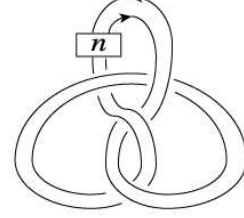
*Proof.*  $K_1$  is a figure eight knot and  $K_2$  is a right handed trefoil knot. By 2(iii),  $M_n(K_1, K_2)$  is diffeomorphic to  $S_{+1}^3(D_-(K_2, n))$ . Therefore  $\lambda(M_n(K_1, K_2))$  is equal to  $\lambda(S_{+1}^3(D_-(K_2, n)))$ . By Lemmas 2.1 and 2.2, we can compute the Casson invariant  $\lambda(S_{+1}^3(D_-(K_2, n)))$  as follows:

FIGURE 2.63.  $K_-$ FIGURE 2.64.  $K_+$ FIGURE 2.65.  $K_0$ 

By Lemma 2.1,  $\lambda'(K_+) - \lambda'(K_-) = lk(K_0)$ . Since  $K_+$  is a trivial knot,  $\lambda'(K_+) = 0$ .  $lk(K_0)$  is  $-n$ . Therefore,  $\lambda'(K_-) = n$ . Then  $\lambda(M_n(K_1, K_2)) = \lambda(S_{+1}^3(D_-(K_2, n))) = \lambda'(K_-) = n$ .  $\square$

2(ix). **The Casson invariant of the fourth row on Theorem 1.1's table.**

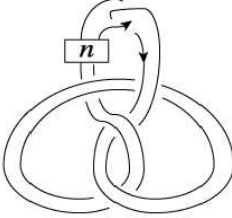
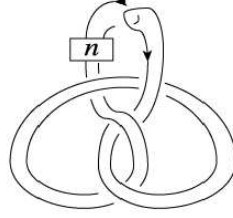
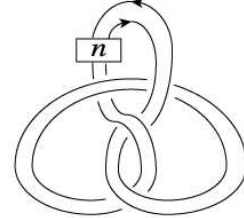
*Proof.*  $K_1$  is a right handed trefoil knot and  $K_2$  is a figure eight knot. By 2(iv),  $M_n(K_1, K_2)$  is diffeomorphic to  $S_{+1}^3(D_+(K_2, n))$ . Therefore  $\lambda(M_n(K_1, K_2))$  is equal to  $\lambda(S_{+1}^3(D_+(K_2, n)))$ . By Lemmas 2.1 and 2.2, we can compute the Casson invariant  $\lambda(S_{+1}^3(D_+(K_2, n)))$  as follows:

FIGURE 2.66.  $K_+$ FIGURE 2.67.  $K_-$ FIGURE 2.68.  $K_0$ 

By Lemma 2.1,  $\lambda'(K_+) - \lambda'(K_-) = lk(K_0)$ . Since  $K_-$  is a trivial knot,  $\lambda'(K_-) = 0$ .  $lk(K_0)$  is  $-n$ . Therefore,  $\lambda'(K_+) = -n$ . Then  $\lambda(M_n(K_1, K_2)) = \lambda(S_{+1}^3(D_+(K_2, n))) = \lambda'(K_+) = -n$ .  $\square$

## 2(x). The Casson invariant of the fifth row on Theorem 1.1's table.

*Proof.*  $K_1$  and  $K_2$  are figure eight knots. By 2(v),  $M_n(K_1, K_2)$  is diffeomorphic to  $S_{+1}^3(D_-(K_2, n))$ . Therefore  $\lambda(M_n(K_1, K_2))$  is equal to  $\lambda(S_{+1}^3(D_-(K_2, n)))$ . By Lemmas 2.1 and 2.2, we can compute the Casson invariant  $\lambda(S_{+1}^3(D_-(K_2, n)))$  as follows:

FIGURE 2.69.  $K_-$ FIGURE 2.70.  $K_+$ FIGURE 2.71.  $K_0$ 

By Lemma 2.1,  $\lambda'(K_+) - \lambda'(K_-) = lk(K_0)$ . Since  $K_+$  is a trivial knot,  $\lambda'(K_+) = 0$ .  $lk(K_0)$  is  $-n$ . Therefore,  $\lambda'(K_-) = n$ . Then  $\lambda(M_n(K_1, K_2)) = \lambda(S_{+1}^3(D_-(K_2, n))) = \lambda'(K_-) = n$ .  $\square$

## 3. Proof of Proposition 1.7

We show that  $V_n^1 \cup_{\partial} (-V_n^2)$  is diffeomorphic to  $\mathbb{CP}^2 \sharp \mathbb{CP}^2$ .

*Proof.* By Kirby Calculus, we will show that the Kirby diagram of  $V_n^1 \cup_{\partial} (-V_n^2)$  is represented by Figure 3.6 :



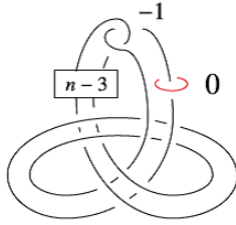


FIGURE 3.1.

$\xrightarrow[\cong]{\partial}$   
 by the same  
 process except for  
 the sign of  
 the framing  
 FIGURE 2.5 - FIGURE 2.17

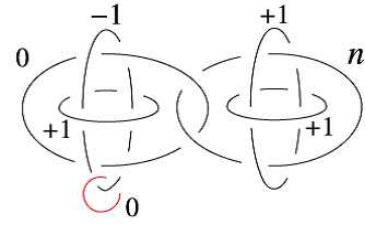


FIGURE 3.2.

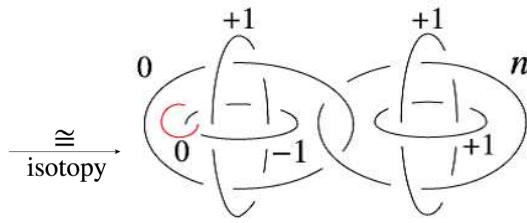


FIGURE 3.3.

$\xrightarrow[\cong]{\partial}$   
 by the same  
 process except for  
 the sign of  
 the framing  
 FIGURE 2.5 - FIGURE 2.9

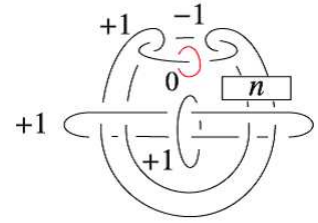


FIGURE 3.4.

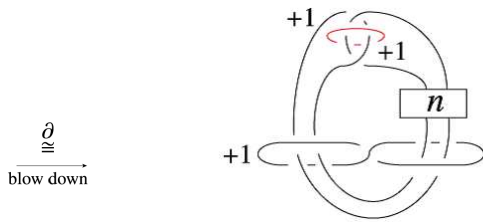
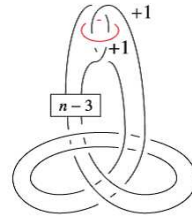
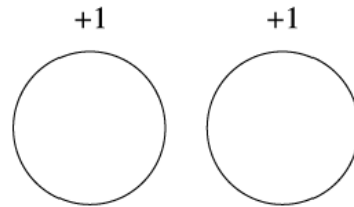


FIGURE 3.5.

FIGURE 3.6.  $V_n^1 \cup_{\partial} (-V_n^2)$ 

$\xrightarrow[\text{handle slide}]{\cong}$

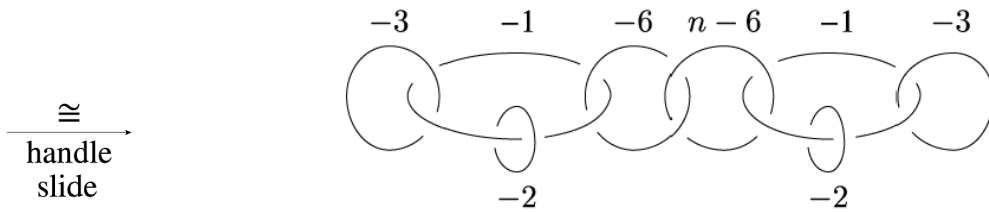
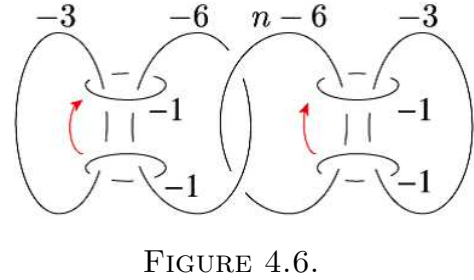
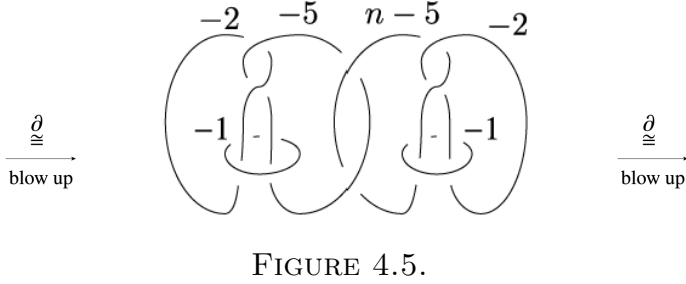
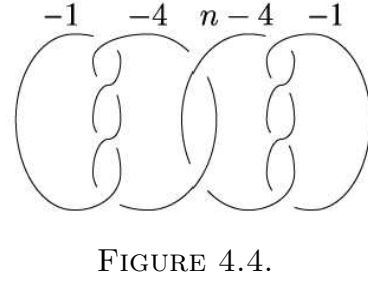
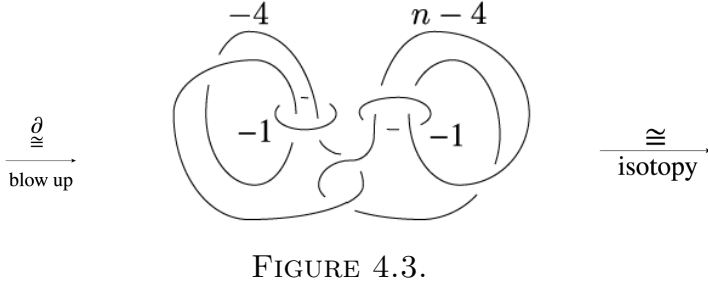
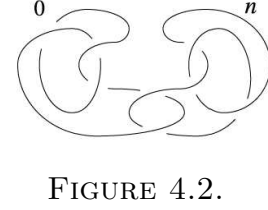
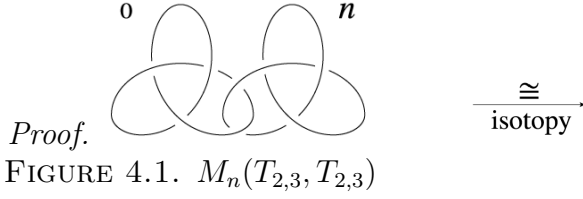
FIGURE 3.7.  $\mathbb{CP}^2 \# \mathbb{CP}^2$ 

□

#### 4. Appendix

**An alternative proof of Corollary 1.2.** By [3], if  $n$  is odd,  $M_n(T_{2,3}, T_{2,3})$  does not bound any contractible 4-manifold. In this Section we will give an alternative proof of this fact. For this purpose, we will prove the following proposition;

**Proposition 4.1.** *The 4-dimensional handlebodies represented by Figures 4.1, 4.8 and 4.9 have the same boundaries.*



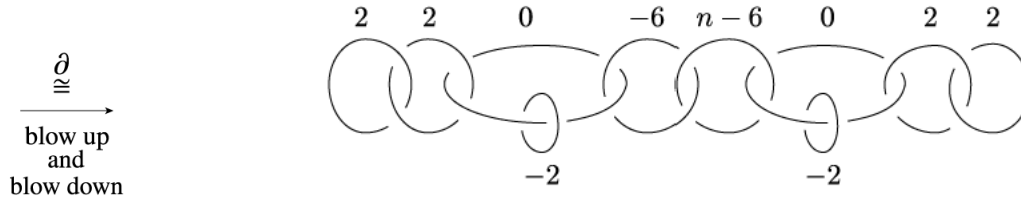


FIGURE 4.8.

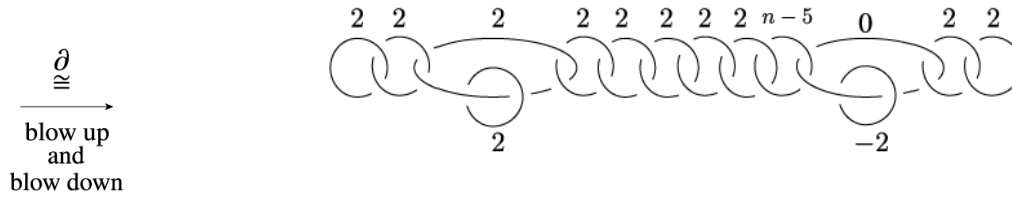
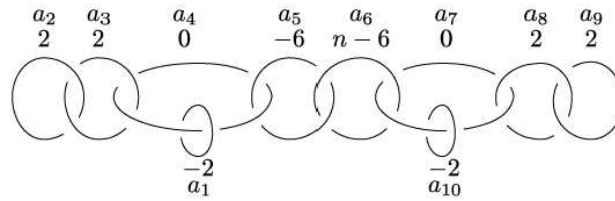


FIGURE 4.9.

□

We give an alternative proof of Corollary 1.2.

*Proof.* Figure 4.8 gives a smooth 4-manifold  $Q_1$  with intersection form  $A$ .

FIGURE 4.10.  $Q_1$ 

$$A = (\alpha_{ij}), \quad \alpha_{ij} = a_i \cdot a_j, \quad 1 \leq i, j \leq 10$$

$$A = \begin{pmatrix} -2 & & & & & & & & & & & & & & \\ & 2 & 1 & & & & & & & & & & & & \\ & & 1 & 2 & 1 & & & & & & & & & & \\ & & & 1 & 0 & 1 & & & & & & & & & \\ & & & & 1 & -6 & 1 & & & & & & & & \\ & & & & & 1 & n-6 & 1 & & & & & & & \\ & & & & & & 1 & 0 & 1 & & & & & & 1 \\ & & & & & & & 1 & 2 & 1 & & & & & \\ & & & & & & & & 1 & 2 & & & & & \\ & & & & & & & & & 1 & & & & & \\ & & & & & & & & & & 1 & & & & -2 \end{pmatrix}$$

We have  $\text{Index}(A) = 0$ . Note that  $A$  is an even type matrix if  $n$  is even.

Figure 4.9 gives a smooth 4-manifold  $Q_2$  with intersection form  $B$ .

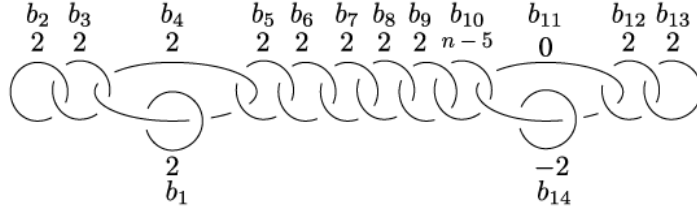


FIGURE 4.11.  $Q_2$

$$B = (\beta_{ij}), \quad \beta_{ij} = b_i \cdot b_j, \quad 1 \leq i, j \leq 14$$

$$B = \begin{pmatrix} 2 & & & & & & & & & & & & & & \\ & 2 & 1 & & & & & & & & & & & & \\ & & 1 & 2 & 1 & & & & & & & & & & \\ & & & 1 & 2 & 1 & & & & & & & & & \\ & & & & 1 & 2 & 1 & & & & & & & & \\ & & & & & 1 & 2 & 1 & & & & & & & \\ & & & & & & 1 & 2 & 1 & & & & & & \\ & & & & & & & 1 & 2 & 1 & & & & & \\ & & & & & & & & 1 & n-5 & 1 & & & & \\ & & & & & & & & & 1 & 0 & 1 & & & 1 \\ & & & & & & & & & & 1 & 2 & 1 & & \\ & & & & & & & & & & & 1 & 2 & & \\ & & & & & & & & & & & & 1 & & -2 \end{pmatrix}$$

We have  $\text{Index}(B) = 8$ . Note that  $B$  is an even type matrix if  $n$  is odd.

By Proposition 4.1, we have the Rohlin invariant  $\mu(M_n(T_{2,3}, T_{2,3}))$  as follows:

$$\mu(M_n(T_{2,3}, T_{2,3})) \equiv \begin{cases} \text{Index}(B) \equiv 1 & (n \text{ is odd}) \\ \text{Index}(A) \equiv 0 & (n \text{ is even}) \end{cases} \pmod{2}$$

Therefore if  $n$  is odd,  $M_n(T_{2,3}, T_{2,3})$  does not bound any contractible 4-manifold.  $\square$

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